

A TRANSFER PRINCIPLE FOR DEVIATIONS PRINCIPLES

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Abstract

This note is not intended for publication. It provides a tool to infer moderate deviations principles for specific random variables from deviations principles for their Hubbard-Stratonovich transforms. This is needed for [2], wherefrom all notation is adopted.

This article's sole purpose is to state and prove the following theorem:

Theorem 1. *Let m be a (local or global) minimum of G and let m be of type k and strength λ .*

(i) *Suppose that*

$$\left(P_{n,\beta}^h \circ \left(\frac{S_n - nm}{n^\alpha} + \frac{W}{n^{\alpha-\frac{1}{2}}} \right)^{-1} \right)_{n \in \mathbb{N}}$$

satisfies for μ -a. e. realization h of \mathbf{h} an MDP with speed $n^{1-2k(1-\alpha)}$ and rate function

$$J(x) := J_{\lambda,k}(x) := \frac{\lambda x^{2k}}{(2k)!}. \quad (1)$$

Then,

$$\left(P_{n,\beta}^{\mathbf{h}} \circ \left(\frac{S_n - nm}{n^\alpha} \right)^{-1} \right)_{n \in \mathbb{N}}$$

satisfies μ -a. s. an MDP with speed $n^{1-2k(1-\alpha)}$ and rate function

$$I(x) := I_{k,\lambda,\beta}(x) := \begin{cases} \frac{x^2}{2\sigma^2}, & \text{if } k = 1, \\ \frac{\lambda x^{2k}}{(2k)!}, & \text{if } k \geq 2, \end{cases} \quad (2)$$

where $\sigma^2 := \lambda^{-1} - \beta^{-1} > 0$.

(ii) *Suppose that*

$$\beta > \frac{2h}{b^2}.$$

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Let c be the supremum of all $x \in (0, (b - \sqrt{2h/\beta})/2]$ such that m is the only minimum of G in $[m - x, m + x]$ and fix $0 < a < c$. Suppose that

$$\left(P_{n,\beta}^h \left(\frac{S_n - nm}{n^\alpha} + \frac{W}{n^{\alpha-\frac{1}{2}}} \in \bullet \mid \frac{S_n - nm}{n^\alpha} + \frac{W}{n^{\alpha-\frac{1}{2}}} \in [-an^{1-\alpha}, an^{1-\alpha}] \right) \right)_{n \in \mathbb{N}}$$

satisfies for μ -a. e. realization h of \mathbf{h} an MDP with speed $n^{1-2k(1-\alpha)}$ and rate function J given by (1). Then,

$$\left(P_{n,\beta}^h \left(\frac{S_n - nm}{n^\alpha} \in \bullet \mid \frac{S_n}{n} \in [m - a, m + a] \right) \right)_{n \in \mathbb{N}}$$

satisfies μ -a. s. an MDP with speed $n^{1-2k(1-\alpha)}$ and rate function I given by (2).

Remark 1. Using Lebesgue's dominated convergence theorem we see

$$G''(x) = \beta - \beta^2 \int_{\mathbb{R}} \frac{1}{\cosh^2(\beta(x+y))} d\nu(y) < \beta.$$

Since for $k = 1$ there exists $x_{\max} \in \mathbb{R}$ such that $\lambda = G''(x_{\max})$, we immediately see $\beta > \lambda$ and, consequently, $\sigma^2 > 0$.

Proof of Theorem 1. Let $X_n := (S_n - nm)/n^\alpha$ and $Y_n := W/n^{\alpha-1/2}$. Choose h such that $P_{n,\beta}^h \circ (X_n + Y_n)^{-1}$ in case (i) resp. $P_{n,\beta}^h(X_n + Y_n \in \bullet \mid X_n + Y_n \in [-an^{1-\alpha}, an^{1-\alpha}])$ in case (ii) satisfy MDPs with speed $n^{2\alpha-1}$ and rate function J . This can be done with probability 1 due to the assumptions. Moreover, note that $P_{n,\beta}^h \circ Y_n^{-1}$ satisfies an MDP with speed $n^{2\alpha-1}$ and rate function $K(x) = \beta x^2/2$ as it can be seen by means of the Gärtner-Ellis Theorem.

ad (i): Let us first consider the case $k > 1$ and see that the influence of the Gaussian random variable vanishes for $n \rightarrow \infty$. Fix $\varepsilon > 0$ and note that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{n^{1-2k(1-\alpha)}} \ln P_{n,\beta}^h(|X_n + Y_n - X_n| > \varepsilon) \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n^{1-2k(1-\alpha)}} \ln P_{n,\beta}^h(|Y_n| > \varepsilon) \\ &= \limsup_{n \rightarrow \infty} \frac{n^{2(k-1)(1-\alpha)}}{n^{2\alpha-1}} \ln P_{n,\beta}^h(|Y_n| > \varepsilon) \\ &= -\infty, \end{aligned}$$

since $2(k-1)(1-\alpha) > 0$, and, by the MDP for $P_{n,\beta}^h \circ Y_n^{-1}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(|Y_n| > \varepsilon) = -K(\varepsilon) < 0.$$

Therefore, $X_n + Y_n$ and X_n are exponentially equivalent on the scale $n^{1-2k(1-\alpha)}$ and, thus, satisfy the same MDP (cf. Theorem 4.2.13 in [1]).

Now consider the case $k = 1$. Note that it suffices to prove

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \geq x) &= -I(x) \text{ for } x \geq 0 \text{ and} \\ \lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \leq x) &= -I(x) \text{ for } x \leq 0 \end{aligned} \tag{3}$$

to gain the full MDP for $P_{n,\beta}^h \circ X_n^{-1}$, i. e.

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \in C) &\leq - \inf_{x \in C} I(x) \text{ for every closed set } C \subset \mathbb{R}, \\ \liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \in O) &\geq - \inf_{x \in O} I(x) \text{ for every open set } O \subset \mathbb{R}. \end{aligned}$$

Indeed, if $0 \in C$, then $\inf_{x \in C} I(x) = 0$ and the upper bound holds trivially as $\ln P_{n,\beta}^h(X_n \in C)$ is always non-positive. On the other hand, if $0 \notin C$, we can define $a := \text{dist}(C, \{0\})$, which is positive as C is closed. Using (3) we obtain the general upper bound

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \in C) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \in (-\infty, -a] \cup [a, \infty)) \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln(P_{n,\beta}^h(X_n \leq -a) + P_{n,\beta}^h(X_n \geq a)) \\ &= \max \left\{ \limsup_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \leq -a), \limsup_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \geq a) \right\} \\ &= -I(a) \\ &= - \inf_{x \in C} I(x) \end{aligned}$$

where we have made use of Lemma 1.2.15 from [1] to derive the last but two line and of I 's monotonicity to derive the last line. To see that also the general lower bound follows from (3), we first note that (3) implies the lower bound for arbitrary balls $B_\varepsilon(x) := \{y \in \mathbb{R} \mid |x - y| < \varepsilon\}$ of radius $\varepsilon > 0$ centered at $x \in \mathbb{R}$:

- First case: $\varepsilon > |x|$

There exists $\delta > 0$ such that $B_\delta(0) \subset B_\varepsilon(x)$ and, consequently,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \in B_\varepsilon(x)) &\geq \liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(|X_n| < \delta) \\ &= 0 \\ &= - \inf_{y \in B_\varepsilon(x)} I(y), \end{aligned}$$

where we have used in the second line that (3) implies $P_{n,\beta}^h(|X_n| < \delta) = 1 - P_{n,\beta}^h(X_n \geq \delta) - P_{n,\beta}^h(X_n \leq -\delta) \rightarrow 1$.

- Second case: $x \geq \varepsilon$

(3) yields for every $\delta > 0$ and n sufficiently large

$$\begin{aligned} P_{n,\beta}^h(X_n \geq x - \varepsilon + \delta) &\geq e^{n^{2\alpha-1}(-I(x-\varepsilon+\delta)-\delta)}, \\ P_{n,\beta}^h(X_n \geq x + \varepsilon) &\leq e^{n^{2\alpha-1}(-I(x+\varepsilon)+\delta)}. \end{aligned}$$

Since I is a continuous function with $I(x + \varepsilon) > I(x - \varepsilon)$ we get

$$-I(x + \varepsilon) + \delta + I(x - \varepsilon + \delta) + \delta < 0$$

for $\delta > 0$ sufficiently small and, therefore,

$$\begin{aligned} &\liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \in B_\varepsilon(x)) \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln(P_{n,\beta}^h(X_n \geq x - \varepsilon + \delta) - P_{n,\beta}^h(X_n \geq x + \varepsilon)) \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln(e^{n^{2\alpha-1}(-I(x-\varepsilon+\delta)-\delta)}(1 - e^{n^{2\alpha-1}(-I(x+\varepsilon)+\delta+I(x-\varepsilon+\delta)+\delta)})) \\ &= -I(x - \varepsilon + \delta) - \delta \end{aligned}$$

for $\delta > 0$ sufficiently small. Taking $\delta \searrow 0$ yields

$$\liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \in B_\varepsilon(x)) \geq -I(x - \varepsilon) = - \inf_{y \in B_\varepsilon(x)} I(y).$$

- Third case: $x \leq -\varepsilon$

Again, (3) yields for every $\delta > 0$ and n sufficiently large

$$\begin{aligned} P_{n,\beta}^h(X_n \leq x + \varepsilon - \delta) &\geq e^{n^{2\alpha-1}(-I(x+\varepsilon-\delta)-\delta)}, \\ P_{n,\beta}^h(X_n \leq x - \varepsilon) &\leq e^{n^{2\alpha-1}(-I(x-\varepsilon)+\delta)}, \end{aligned}$$

which implies

$$\begin{aligned} &\liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \in B_\varepsilon(x)) \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln(P_{n,\beta}^h(X_n \leq x + \varepsilon - \delta) - P_{n,\beta}^h(X_n \leq x - \varepsilon)) \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln(e^{n^{2\alpha-1}(-I(x+\varepsilon-\delta)-\delta)}(1 - e^{n^{2\alpha-1}(-I(x-\varepsilon)+\delta+I(x+\varepsilon-\delta)+\delta)})). \end{aligned}$$

Since $I(x - \varepsilon) > I(x + \varepsilon)$ the continuity of I yields

$$-I(x - \varepsilon) + \delta + I(x + \varepsilon - \delta) + \delta < 0$$

for $\delta > 0$ sufficiently small and, consequently,

$$\liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \in B_\varepsilon(x)) \geq -I(x + \varepsilon - \delta) - \delta$$

for $\delta > 0$ sufficiently small. Once again, taking $\delta \searrow 0$ yields

$$\liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \in B_\varepsilon(x)) \geq -I(x + \varepsilon) = - \inf_{y \in B_\varepsilon(x)} I(y).$$

The lower bound for open balls already gives the lower bound for arbitrary open sets. In fact, fix $G \subset \mathbb{R}$ open and let x be an element of G (the case $G = \emptyset$ holds trivially). Then, there exists $\varepsilon > 0$ s. t. $B_\varepsilon(x) \subset G$ and therefore

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \in G) &\geq \liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \in B_\varepsilon(x)) \\ &\geq - \inf_{y \in B_\varepsilon(x)} I(y) \\ &\geq -I(x) \end{aligned}$$

for every $x \in G$. Taking the supremum over all $x \in G$ gives the desired lower bound. In a nutshell, we have seen that (3) yields the desired MDP and, therefore, we are left with a proof of (3), which we start with a first observation:

$$\begin{aligned} I(x) &= -\frac{\lambda\beta}{2(\beta-\lambda)}x^2 \\ &= -\frac{\lambda\beta^2}{2(\beta-\lambda)^2}x^2 + \frac{\beta\lambda^2}{2(\beta-\lambda)^2}x^2 \\ &= -\frac{\lambda}{2}x_0^2 + \frac{\beta}{2}(x_0 - x)^2 \\ &= -J(x_0) + K(x_0 - x), \end{aligned} \tag{4}$$

where $x_0 := \frac{\beta}{\beta-\lambda}x$.

Upper bounds in (3):

Let us first consider the case $x \geq 0$. Since X_n and Y_n are independent, we have

$$P_{n,\beta}^h(X_n \geq x)P_{n,\beta}^h(Y_n \geq x_0 - x) \leq P_{n,\beta}^h(X_n + Y_n \geq x_0)$$

respectively

$$P_{n,\beta}^h(X_n \geq x) \leq \frac{P_{n,\beta}^h(X_n + Y_n \geq x_0)}{P_{n,\beta}^h(Y_n \geq x_0 - x)}.$$

Using the MDPs for $P_{n,\beta}^h \circ Y_n^{-1}$ and $P_{n,\beta}^h \circ (X_n + Y_n)^{-1}$, we see

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \geq x) &\leq \limsup_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln \frac{P_{n,\beta}^h(X_n + Y_n \geq x_0)}{P_{n,\beta}^h(Y_n \geq x_0 - x)} \\ &= -J(x_0) + K(x_0 - x) \\ &= I(x) \end{aligned}$$

where we have used the introductory observation (4). In the remaining case $x \leq 0$, we have

$$P_{n,\beta}^h(X_n \leq x) \leq \frac{P_{n,\beta}^h(X_n + Y_n \leq x_0)}{P_{n,\beta}^h(Y_n \leq x_0 - x)},$$

and with the same arguments as before we prove

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \leq x) \leq I(x).$$

Lower bounds in (3):

Again, we first consider the case $x \geq 0$. Using (4) and the continuity of J it suffices to show

$$\liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \geq x) \geq -J(x_0 + \varepsilon) + K(x_0 - x)$$

for every $\varepsilon > 0$ or, equivalently,

$$\liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln \frac{P_{n,\beta}^h(X_n \geq x)P_{n,\beta}^h(Y_n \geq x_0 - x)}{P_{n,\beta}^h(X_n + Y_n \geq x_0 + \varepsilon)} \geq 0 \quad (5)$$

for all $\varepsilon > 0$, where we have made use of the MDPs for $P_{n,\beta}^h \circ Y_n^{-1}$ and $P_{n,\beta}^h \circ (X_n + Y_n)^{-1}$. Fix $\varepsilon > 0$ and note that since

$$\begin{aligned} &\frac{P_{n,\beta}^h(X_n \geq x)P_{n,\beta}^h(Y_n \geq x_0 - x)}{P_{n,\beta}^h(X_n + Y_n \geq x_0 + \varepsilon)} \\ &\geq P_{n,\beta}^h(X_n \geq x, Y_n \geq x_0 - x \mid X_n + Y_n \geq x_0 + \varepsilon) \\ &= 1 - P_{n,\beta}^h(X_n < x \mid X_n + Y_n \geq x_0 + \varepsilon) - P_{n,\beta}^h(Y_n < x_0 - x \mid X_n + Y_n \geq x_0 + \varepsilon) \end{aligned}$$

(5) follows once we have proved

$$P_{n,\beta}^h(X_n < x \mid X_n + Y_n \geq x_0 + \varepsilon) = o(1), \quad (6)$$

$$P_{n,\beta}^h(Y_n < x_0 - x \mid X_n + Y_n \geq x_0 + \varepsilon) = o(1). \quad (7)$$

We start with a proof of (6). It is straightforward to see that

$$\begin{aligned}
& P_{n,\beta}^h(X_n < x_0(1 - \sqrt{\lambda/\beta}) | X_n + Y_n \geq x_0 + \varepsilon) \\
&= \frac{P_{n,\beta}^h(X_n < x_0 - x_0\sqrt{\lambda/\beta}, X_n + Y_n \geq x_0 + \varepsilon)}{P_{n,\beta}^h(X_n + Y_n \geq x_0 + \varepsilon)} \\
&\leq \frac{P_{n,\beta}^h(Y_n \geq x_0\sqrt{\lambda/\beta} + \varepsilon)}{P_{n,\beta}^h(X_n + Y_n \geq x_0 + \varepsilon)},
\end{aligned}$$

which again can be bounded using the MDPs for $P_{n,\beta}^h \circ Y_n^{-1}$ and $P_{n,\beta}^h \circ (X_n + Y_n)^{-1}$, which yield

$$\begin{aligned}
P_{n,\beta}^h(Y_n \geq x_0\sqrt{\lambda/\beta} + \varepsilon) &\leq e^{-n^{2\alpha-1}(K(x_0\sqrt{\lambda/\beta} + \varepsilon) - \frac{\beta-\lambda}{4}\varepsilon^2)}, \\
P_{n,\beta}^h(X_n + Y_n \geq x_0 + \varepsilon) &\geq e^{-n^{2\alpha-1}(J(x_0 + \varepsilon) + \frac{\beta-\lambda}{4}\varepsilon^2)}
\end{aligned}$$

for n sufficiently large. Consequently, for n sufficiently large

$$\begin{aligned}
& P_{n,\beta}^h(X_n < x_0(1 - \sqrt{\lambda/\beta}) | X_n + Y_n \geq x_0 + \varepsilon) \\
&\leq e^{-n^{2\alpha-1}(K(x_0\sqrt{\lambda/\beta} + \varepsilon) - \frac{\beta-\lambda}{4}\varepsilon^2 - J(x_0 + \varepsilon) - \frac{\beta-\lambda}{4}\varepsilon^2)} \\
&= e^{-n^{2\alpha-1}(\frac{\beta}{2}(x_0\sqrt{\lambda/\beta} + \varepsilon)^2 - \frac{\lambda}{2}(x_0 + \varepsilon)^2 - \frac{\beta-\lambda}{2}\varepsilon^2)} \\
&= e^{-n^{2\alpha-1}\varepsilon x_0\sqrt{\lambda}(\sqrt{\beta} - \sqrt{\lambda})} \\
&= o(1).
\end{aligned}$$

If $x = 0$, then $x_0(1 - \sqrt{\lambda/\beta}) = x$ and we are done proving (6). Otherwise, $x_0(1 - \sqrt{\lambda/\beta}) < x$, and we need to show that $P_{n,\beta}^h(x_0(1 - \sqrt{\lambda/\beta}) \leq X_n < x | X_n + Y_n \geq x_0 + \varepsilon)$ is a zero sequence. Let us divide, to that end, the interval $[x_0(1 - \sqrt{\lambda/\beta}), x]$ into $M \in \mathbb{N}$ subintervals, each of length \hat{x}/M where $\hat{x} := x - x_0(1 - \sqrt{\lambda/\beta}) > 0$. We get

$$\begin{aligned}
& P_{n,\beta}^h(x_0(1 - \sqrt{\lambda/\beta}) \leq X_n < x | X_n + Y_n \geq x_0 + \varepsilon) \\
&= \sum_{i=1}^M \frac{P_{n,\beta}^h(x_0(1 - \sqrt{\lambda/\beta}) + \frac{i-1}{M}\hat{x} \leq X_n < x_0(1 - \sqrt{\lambda/\beta}) + \frac{i}{M}\hat{x}, X_n + Y_n \geq x_0 + \varepsilon)}{P_{n,\beta}^h(X_n + Y_n \geq x_0 + \varepsilon)} \\
&\leq \sum_{i=1}^M \frac{P_{n,\beta}^h(x_0(1 - \sqrt{\lambda/\beta}) + \frac{i-1}{M}\hat{x} \leq X_n) P_{n,\beta}^h(Y_n \geq x_0 + \varepsilon - x_0(1 - \sqrt{\lambda/\beta}) - \frac{i}{M}\hat{x})}{P_{n,\beta}^h(X_n + Y_n \geq x_0 + \varepsilon)} \\
&= \sum_{i=1}^M \frac{P_{n,\beta}^h(X_n \geq x_0(1 - \sqrt{\lambda/\beta}) + \frac{i-1}{M}\hat{x}) P_{n,\beta}^h(Y_n \geq x_0\sqrt{\lambda/\beta} + \varepsilon - \frac{i}{M}\hat{x})}{P_{n,\beta}^h(X_n + Y_n \geq x_0 + \varepsilon)}. \tag{8}
\end{aligned}$$

Using the upper bound for $\limsup_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \geq \cdot)$, which we have obtained before, and the MDPs for $P_{n,\beta}^h \circ Y_n^{-1}$ and $P_{n,\beta}^h \circ (X_n + Y_n)^{-1}$, we get for every $1 \leq i \leq M$ and n sufficiently large

$$\begin{aligned}
P_{n,\beta}^h(X_n \geq x_0(1 - \sqrt{\lambda/\beta}) + (i-1)\hat{x}/M) &\leq e^{-n^{2\alpha-1}(I(x_0(1 - \sqrt{\lambda/\beta}) + \frac{i-1}{M}\hat{x}) - \delta/3)}, \\
P_{n,\beta}^h(Y_n \geq x_0\sqrt{\lambda/\beta} + \varepsilon - i\hat{x}/M) &\leq e^{-n^{2\alpha-1}(K(x_0\sqrt{\lambda/\beta} + \varepsilon - \frac{i}{M}\hat{x}) - \delta/3)}, \\
P_{n,\beta}^h(X_n + Y_n \geq x_0 + \varepsilon) &\geq e^{-n^{2\alpha-1}(J(x_0 + \varepsilon) + \delta/3)},
\end{aligned}$$

where $\delta := (\beta - \lambda)\varepsilon^2/4 > 0$. Inserting these estimates in (8) yields

$$\begin{aligned}
& P_{n,\beta}^h(x_0(1 - \sqrt{\lambda/\beta}) \leq X_n < x | X_n + Y_n \geq x_0 + \varepsilon) \\
&\leq \sum_{i=1}^M e^{-n^{2\alpha-1}(I(x_0(1 - \sqrt{\lambda/\beta}) + \frac{i-1}{M}\hat{x}) + K(x_0\sqrt{\lambda/\beta} + \varepsilon - \frac{i}{M}\hat{x}) - J(x_0 + \varepsilon) - \delta)} \tag{9}
\end{aligned}$$

To find an upper bound for (9), we need to find the dominating summand. Note to this purpose that the function

$$f : \mathbb{R} \rightarrow \mathbb{R},$$

$$z \mapsto I(x_0(1 - \sqrt{\lambda/\beta}) - \hat{x}/M + z\hat{x}) + K(x_0\sqrt{\lambda/\beta} + \varepsilon - z\hat{x})$$

is decreasing on $[0,1]$:

$$\begin{aligned} & \sup_{z \in [0,1]} f'(z) \\ &= \sup_{z \in [0,1]} \left\{ \frac{\hat{x}}{\sigma^2} (x_0(1 - \sqrt{\lambda/\beta}) - \hat{x}/M + z\hat{x}) - \beta\hat{x}(x_0\sqrt{\lambda/\beta} + \varepsilon - z\hat{x}) \right\} \\ &= \frac{\hat{x}}{\sigma^2} (x_0(1 - \sqrt{\lambda/\beta}) - \hat{x}/M + \hat{x}) - \beta\hat{x}(x_0\sqrt{\lambda/\beta} + \varepsilon - \hat{x}) \\ &< \hat{x} \left(\frac{x_0(1 - \sqrt{\lambda/\beta})}{\sigma^2} + \frac{\hat{x}}{\sigma^2} - \beta x_0\sqrt{\lambda/\beta} + \beta\hat{x} \right) \\ &= \hat{x} \left(\frac{x}{\sigma^2} + \beta x - \beta x_0 \right) \\ &= \hat{x} x \left(\frac{\beta\lambda}{\beta - \lambda} + \beta - \frac{\beta^2}{\beta - \lambda} \right) \\ &= 0. \end{aligned}$$

This means that in (9) the summand for $i = M$ is dominating and we get

$$\begin{aligned} & P_{n,\beta}^h(x_0(1 - \sqrt{\lambda/\beta}) \leq X_n < x | X_n + Y_n \geq x_0 + \varepsilon) \\ &\leq M e^{-n^{2\alpha-1}(I(x_0(1 - \sqrt{\lambda/\beta}) + \frac{M-1}{M}\hat{x}) + K(x_0\sqrt{\lambda/\beta} + \varepsilon - \hat{x}) - J(x_0 + \varepsilon) - \delta)} \\ &= M e^{-n^{2\alpha-1}\left(\frac{1}{2\sigma^2}(x - \hat{x}/M)^2 + \frac{\beta}{2}(x_0 - x + \varepsilon)^2 - \frac{\lambda}{2}(x_0 + \varepsilon)^2 - \delta\right)} \\ &= M e^{-n^{2\alpha-1}\left(\delta - \frac{x\hat{x}}{\sigma^2 M} + \frac{\hat{x}^2}{2\sigma^2 M^2}\right)}, \end{aligned}$$

which is converging to 0 if M is sufficiently large. That finishes the proof of (6).

To complete the proof of the upper bound in (3) for $x \geq 0$ we need to prove (7). As a start we note that

$$\begin{aligned} & P_{n,\beta}^h(Y_n < x_0(1 - \sqrt{\lambda\sigma^2}) | X_n + Y_n \geq x_0 + \varepsilon) \\ &\leq \frac{P_{n,\beta}^h(X_n \geq x_0\sqrt{\lambda\sigma^2} + \varepsilon)}{P_{n,\beta}^h(X_n + Y_n \geq x_0 + \varepsilon)} \\ &\leq e^{-n^{2\alpha-1}(I(x_0\sqrt{\lambda\sigma^2} + \varepsilon) - J(x_0 + \varepsilon) - \frac{\lambda^2}{2(\beta - \lambda)}\varepsilon^2)} \\ &= e^{-n^{2\alpha-1}\varepsilon x_0\lambda(\sqrt{\beta/(\beta - \lambda)} - 1)} \end{aligned}$$

for n sufficiently large, which converges to 0. Therein, we have used the MDP for $P_{n,\beta}^h \circ (X_n + Y_n)^{-1}$ and the upper bound for $\limsup_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln P_{n,\beta}^h(X_n \geq \cdot)$ obtained before. If $x = 0$, then $x_0(1 - \sqrt{\lambda\sigma^2}) = x_0 - x$ and we have proved (7). Otherwise, $x_0(1 - \sqrt{\lambda\sigma^2}) < x_0 - x$ and we need to show that

$$P_{n,\beta}^h(x_0(1 - \sqrt{\lambda\sigma^2}) \leq Y_n < x_0 - x | X_n + Y_n \geq x_0 + \varepsilon) = o(1).$$

We divide the interval $[x_0(1 - \sqrt{\lambda\sigma^2}), x_0 - x]$ into $M \in \mathbb{N}$ subintervals, each of length

\tilde{x}/M where $\tilde{x} := x_0\sqrt{\lambda\sigma^2} - x > 0$. With the same ideas as before we show

$$\begin{aligned}
& P_{n,\beta}^h(x_0(1 - \sqrt{\lambda\sigma^2}) \leq Y_n < x_0 - x | X_n + Y_n \geq x_0 + \varepsilon) \\
& \leq \sum_{i=1}^M \frac{P_{n,\beta}^h(X_n \geq x_0\sqrt{\lambda\sigma^2} + \varepsilon - \frac{i}{M}\tilde{x}) P_{n,\beta}^h(Y_n \geq x_0(1 - \sqrt{\lambda\sigma^2}) + \frac{i-1}{M}\tilde{x})}{P_{n,\beta}^h(X_n + Y_n \geq x_0 + \varepsilon)} \\
& \leq \sum_{i=1}^M e^{-n^{2\alpha-1} \left(I(x_0\sqrt{\lambda\sigma^2} + \varepsilon - \frac{i}{M}\tilde{x}) + K(x_0(1 - \sqrt{\lambda\sigma^2}) + \frac{i-1}{M}\tilde{x}) - J(x_0 + \varepsilon) - \frac{\lambda^2}{4(\beta-\lambda)}\varepsilon^2 \right)} \\
& \leq M e^{-n^{2\alpha-1} \left(I(x_0\sqrt{\lambda\sigma^2} + \varepsilon - \tilde{x}) + K(x_0(1 - \sqrt{\lambda\sigma^2}) + \frac{M-1}{M}\tilde{x}) - J(x_0 + \varepsilon) - \frac{\lambda^2}{4(\beta-\lambda)}\varepsilon^2 \right)} \\
& = M e^{-n^{2\alpha-1} \left(\frac{\lambda^2}{4(\beta-\lambda)}\varepsilon^2 - \frac{\beta(x_0-x)\tilde{x}}{M} + \frac{\beta\tilde{x}^2}{2M^2} \right)}
\end{aligned}$$

for n sufficiently large, which again converges to 0 for M sufficiently large. This ends the proof of (7) yields the lower bound in (3) for $x \geq 0$.

We are left to prove the lower bound in equation (3) for $x \leq 0$. With the same arguments like the ones used in the case $x \geq 0$ it suffices to show

$$\liminf_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \ln \frac{P_{n,\beta}^h(X_n \leq x) P_{n,\beta}^h(Y_n \leq x_0 - x)}{P_{n,\beta}^h(X_n + Y_n \leq x_0 - \varepsilon)} \geq 0$$

for all $\varepsilon > 0$. This follows from

$$\begin{aligned}
P_{n,\beta}^h(X_n > x | X_n + Y_n \leq x_0 - \varepsilon) &= o(1), \\
P_{n,\beta}^h(Y_n > x_0 - x | X_n + Y_n \leq x_0 - \varepsilon) &= o(1).
\end{aligned}$$

Again, with the same arguments as before we show

$$\begin{aligned}
P_{n,\beta}^h(X_n > x_0(1 - \sqrt{\lambda/\beta}) | X_n + Y_n \leq x_0 - \varepsilon) &\leq e^{-n^{2\alpha-1} \varepsilon x_0 (\sqrt{\lambda\beta} - \lambda)} \text{ and} \\
P_{n,\beta}^h(Y_n > x_0(1 - \sqrt{\lambda\sigma^2}) | X_n + Y_n \leq x_0 - \varepsilon) &\leq e^{-n^{2\alpha-1} \varepsilon x_0 \lambda (\sqrt{\beta/(\beta-\lambda)} - 1)}
\end{aligned}$$

for n sufficiently large. Thus, it is left to show

$$\begin{aligned}
P_{n,\beta}^h(x < X_n \leq x_0(1 - \sqrt{\lambda/\beta}) | X_n + Y_n \leq x_0 - \varepsilon) &= o(1) \text{ and} \\
P_{n,\beta}^h(x_0 - x < Y_n \leq x_0(1 - \sqrt{\lambda\sigma^2}) | X_n + Y_n \leq x_0 - \varepsilon) &= o(1).
\end{aligned}$$

To that end, we divide the intervals $(x, x_0(1 - \sqrt{\lambda/\beta})]$ resp. $(x_0 - x, x_0(1 - \sqrt{\lambda\sigma^2})]$ into $M \in \mathbb{N}$ subintervals, each of length \hat{x}/M resp. \tilde{x}/M where $\hat{x} = x_0(1 - \sqrt{\lambda/\beta}) - x$ and $\tilde{x} = x - x_0\sqrt{\lambda\sigma^2}$. Following the lines of the previous case, we get for n sufficiently large

$$\begin{aligned}
& P_{n,\beta}^h(x < X_n \leq x_0(1 - \sqrt{\lambda/\beta}) | X_n + Y_n \leq x_0 - \varepsilon) \\
& \leq M e^{-n^{2\alpha-1} \left(\frac{(\beta-\lambda)\varepsilon^2}{4} + \frac{x\hat{x}}{\sigma^2 M} + \frac{\hat{x}^2}{2\sigma^2 M^2} \right)}
\end{aligned}$$

and

$$\begin{aligned}
& P_{n,\beta}^h(x_0 - x < Y_n \leq x_0(1 - \sqrt{\lambda\sigma^2}) | X_n + Y_n \leq x_0 - \varepsilon) \\
& \leq M e^{-n^{2\alpha-1} \left(\frac{\lambda^2}{4(\beta-\lambda)}\varepsilon^2 + \frac{\beta(x_0-x)\tilde{x}}{M} + \frac{\beta\tilde{x}^2}{2M^2} \right)},
\end{aligned}$$

where we now have used that the dominant terms are the ones for $i = 1$, i. e.

$$P_{n,\beta}^h(x < X_n \leq x + \hat{x}/M | X_n + Y_n \leq x_0 - \varepsilon) \text{ resp.} \\ P_{n,\beta}^h(x_0 - x < Y_n \leq x_0 - x + \tilde{x}/M | X_n + Y_n \leq x_0 - \varepsilon).$$

That finishes the proof of the lower bound in equation (3) and thus all in all part (i).

ad (ii): Let $B_n := [-an^{1-\alpha}, an^{1-\alpha}]$ and choose the realization h of \mathbf{h} such that not only the MDP for $P_{n,\beta}^h(X_n + Y_n \in \bullet | X_n + Y_n \in B_n)$ holds, but also that

$$P_{n,\beta}^h\left(\frac{S_n}{n} \in \bullet\right)$$

satisfies an LDP with speed n and rate function $I_\beta^\nu(x) = \sup_{y \in \mathbb{R}} \{G(y) - \frac{\beta}{2}(x - y)^2\} - \inf_{w \in \mathbb{R}} G(w)$. This can be done with probability 1 due to [3]. Using the Gärtner-Ellis Theorem, we see that

$$P_{n,\beta}^h\left(\frac{W}{\sqrt{n}} \in \bullet\right)$$

satisfies an LDP with speed n and rate function $K(x) = \beta x^2/2$. Thus, a use of the contraction principle yields (cf. Exercise 4.2.7 in [1]) that

$$\left(P_{n,\beta}^h\left(\frac{S_n}{n} + \frac{W}{\sqrt{n}} \in \bullet\right)\right)_{n \in \mathbb{N}}$$

satisfies an LDP with speed n and rate function N given by

$$\begin{aligned} N(x) &= \inf_{\substack{(y,z) \in \mathbb{R}^2: \\ y+z=x}} (K(y) + I_\beta^\nu(z)) \\ &= \inf_{z \in \mathbb{R}} \left(\frac{\beta}{2}(x - z)^2 + \sup_{w \in \mathbb{R}} \left(G(w) - \frac{\beta}{2}(w - z)^2 \right) \right) - \inf_{w \in \mathbb{R}} G(w) \\ &= \frac{\beta}{2}x^2 + \inf_{z \in \mathbb{R}} \left(-\beta xz + \sup_{w \in \mathbb{R}} \left(G(w) - \frac{\beta}{2}w^2 + \beta wz \right) \right) - \inf_{w \in \mathbb{R}} G(w) \\ &= \frac{\beta}{2}x^2 - \sup_{z \in \mathbb{R}} \left(xz - \sup_{w \in \mathbb{R}} \left(wz - \int_{\mathbb{R}} \ln \cosh[\beta(w + h)] d\nu(h) \right) \right) - \inf_{w \in \mathbb{R}} G(w) \\ &= G(x) - \inf_{w \in \mathbb{R}} G(w) \end{aligned}$$

where we have used the duality lemma for Legendre-Fenchel transforms (cf. Lemma 4.5.8 in [1]) to derive the last line. Since X_n and Y_n are *not* independent under the measure $P_{n,\beta}^h(\bullet | X_n + Y_n \in B_n)$ we cannot proceed as in part (i). Instead, we start by showing that the MDP for $P_{n,\beta}^h(X_n + Y_n \in \bullet | X_n + Y_n \in B_n)$ transfers to the same MDP for

$$P_{n,\beta}^h(X_n + Y_n \in \bullet | X_n \in B_n).$$

In order to do this, we show that the two sequences are exponentially equivalent on the scale $n^{1-2k(1-\alpha)}$, i. e.

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{1-2k(1-\alpha)}} \log \rho_n = -\infty \quad (10)$$

where

$$\rho_n := \sup_{B \in \mathcal{B}(\mathbb{R})} \{P_{n,\beta}^h(X_n + Y_n \in B | X_n \in B_n) - P_{n,\beta}^h(X_n + Y_n \in B | X_n + Y_n \in B_n)\}.$$

Note that for every $B \in \mathcal{B}(\mathbb{R})$

$$\begin{aligned}
& P_{n,\beta}^h(X_n + Y_n \in B | X_n \in B_n) - P_{n,\beta}^h(X_n + Y_n \in B | X_n + Y_n \in B_n) \\
& \leq P_{n,\beta}^h(|Y_n| > n^{(1-\alpha)/2}) + P_{n,\beta}^h(X_n + Y_n \in B, |Y_n| \leq n^{(1-\alpha)/2} | X_n \in B_n) \\
& \quad - P_{n,\beta}^h(X_n + Y_n \in B | X_n + Y_n \in B_n) \\
& = P_{n,\beta}^h(|Y_n| > n^{(1-\alpha)/2}) + \left(\frac{1}{P_{n,\beta}^h(X_n \in B_n)} - \frac{1}{P_{n,\beta}^h(X_n + Y_n \in B_n)} \right) \\
& \quad \times P_{n,\beta}^h(X_n + Y_n \in B, X_n \in B_n, |Y_n| \leq n^{(1-\alpha)/2}) \\
& \quad + \frac{P_{n,\beta}^h(X_n + Y_n \in B, X_n \in B_n, |Y_n| \leq n^{(1-\alpha)/2}) - P_{n,\beta}^h(X_n + Y_n \in B \cap B_n)}{P_{n,\beta}^h(X_n + Y_n \in B_n)}
\end{aligned}$$

and consequently ρ_n is bounded by

$$\begin{aligned}
& P_{n,\beta}^h(|Y_n| > n^{(1-\alpha)/2}) + \frac{P_{n,\beta}^h(X_n + Y_n \in B_n) - P_{n,\beta}^h(X_n \in B_n)}{P_{n,\beta}^h(X_n + Y_n \in B_n)} \vee 0 \\
& + \frac{P_{n,\beta}^h(X_n + Y_n \in [-an^{1-\alpha} - n^{(1-\alpha)/2}, -an^{1-\alpha}] \cup [an^{1-\alpha}, an^{1-\alpha} + n^{(1-\alpha)/2}])}{P_{n,\beta}^h(X_n + Y_n \in B_n)}.
\end{aligned}$$

Using Lemma 1.2.15 from [1] (10) follows from proving that each of the three summands converges to $-\infty$ on a logarithmic scale of order $n^{1-2k(1-\alpha)}$.

First,

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{1-2k(1-\alpha)}} \log P_{n,\beta}^h(|Y_n| > n^{(1-\alpha)/2}) = -\infty$$

follows immediately from the standard estimate

$$P(Z > x) \leq \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

for a standard Gaussian $Z, x > 0$.

Second, with $\delta = b - c > 0$ it is

$$\begin{aligned}
& \frac{P_{n,\beta}^h(X_n + Y_n \in B_n) - P_{n,\beta}^h(X_n \in B_n)}{P_{n,\beta}^h(X_n + Y_n \in B_n)} \vee 0 \\
& = \frac{P_{n,\beta}^h(S_n/n + W/\sqrt{n} \in [m - a, m + a]) - P_{n,\beta}^h(S_n/n \in [m - a, m + a])}{P_{n,\beta}^h(S_n/n + W/\sqrt{n} \in [m - a, m + a])} \vee 0 \\
& \leq \frac{P_{n,\beta}^h(S_n/n \in [m - a - \delta, m - a] \cup [m + a, m + a + \delta])}{P_{n,\beta}^h(S_n/n + W/\sqrt{n} \in [m - a, m + a])} \\
& \quad + \frac{P_{n,\beta}^h(|W/\sqrt{n}| > \delta)}{P_{n,\beta}^h(S_n/n + W/\sqrt{n} \in [m - a, m + a])}.
\end{aligned}$$

Using Lemma 1.2.15 in [1] we can again consider the two terms separately. We find

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{P_{n,\beta}^h(S_n/n \in [m - a - \delta, m - a] \cup [m + a, m + a + \delta])}{P_{n,\beta}^h(S_n/n + W/\sqrt{n} \in [m - a, m + a])} \\
& = - \inf_{x \in [m - a - \delta, m - a] \cup [m + a, m + a + \delta]} I_\beta^\nu(x) + \inf_{x \in [m - a, m + a]} N(x) \\
& \leq - \inf_{x \in [m - a - \delta, m - a] \cup [m + a, m + a + \delta]} G(x) + G(m) \\
& < 0
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{P_{n,\beta}^h(|W/\sqrt{n}| > \delta)}{P_{n,\beta}^h(S_n/n + W/\sqrt{n} \in [m-a, m+a])} \\
&= -\frac{\beta\delta^2}{2} + \inf_{x \in [m-a, m+a]} G(x) - \inf_{x \in \mathbb{R}} G(x) \\
&= -\frac{\beta\delta^2}{2} + h \\
&< 0.
\end{aligned}$$

Consequently,

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{1-2k(1-\alpha)}} \log \frac{P_{n,\beta}^h(X_n + Y_n \in B_n) - P_{n,\beta}^h(X_n \in B_n)}{P_{n,\beta}^h(X_n + Y_n \in B_n)} \vee 0 = -\infty.$$

Finally, since m is the only minimum of G in $[m-a, m+a]$ we can choose $\tilde{a} > a$ such that m is also the only minimum of G in $[m-\tilde{a}, m+\tilde{a}]$. Note that

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_{n,\beta}^h(X_n + Y_n \in [-an^{1-\alpha} - n^{(1-\alpha)/2}, -an^{1-\alpha}] \\
& \quad \cup [an^{1-\alpha}, an^{1-\alpha} + n^{(1-\alpha)/2}]) \\
& \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_{n,\beta}^h(S_n/n + W/\sqrt{n} \in [m-\tilde{a}, m-a] \cup [m+a, m+\tilde{a}]) \\
& = - \inf_{x \in [m-\tilde{a}, m-a] \cup [m+a, m+\tilde{a}]} G(x) + \inf_{x \in \mathbb{R}} G(x) \\
& < -G(m) + \inf_{x \in \mathbb{R}} G(x) \\
& = - \inf_{x \in [m-a, m+a]} G(x) + \inf_{x \in \mathbb{R}} G(x) \\
& = \lim_{n \rightarrow \infty} \frac{1}{n} \log P_{n,\beta}^h(S_n/n + W/\sqrt{n} \in [m-a, m+a]) \\
& = \lim_{n \rightarrow \infty} \frac{1}{n} \log P_{n,\beta}^h(X_n + Y_n \in B_n)
\end{aligned}$$

and therefore (10) follows.

Now that we know that $P_{n,\beta}^h(X_n + Y_n \in \bullet | X_n \in B_n)$ satisfies an MDP with speed $n^{1-2k(1-\alpha)}$ and rate function J it is straightforward to prove the MDP for $P_{n,\beta}^h(X_n \in \bullet | X_n \in B_n)$. Since X_n and Y_n are independent under the measure $P_{n,\beta}^h(\bullet | X_n \in B_n)$, the proof can be finished completely analogously to the proof of part (i) with $P_{n,\beta}^h$ replaced by $P_{n,\beta}^h(\bullet | X_n \in B_n)$. \square

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